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Two Ways to Construct a Smooth Piecewise Rational Curve

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Two approaches to constructing piecewise rational curves are compared and contrasted. One involves projecting a piecewise polynomial curve whose polynomial segments are joined according to given totally positive connection matrices. The other, which is discussed in more detail, constructs the Bézier points of the rational segments by successively cutting corners of a given control polygon. The smoothness and total positivity properties of the resulting curves are also discussed.

1. INTRODUCTION

Two of the many seminal ideas which I. J. Schoenberg introduced were the B-spline basis for spline functions [18] and the variation diminishing properties for shape preserving approximation [19]. These ideas have blossomed with subsequent work by Schoenberg and many other authors and have important applications which include the construction of a parametric piecewise polynomial (or more generally piecewise rational) curve whose shape can be determined and manipulated by a sequence of control points.

In this paper we compare two approaches to the construction of such curves. What we call the "analytic approach" was developed latterly by Dyn and Micchelli in [5] in which the curve segments are joined by matching their derivatives by totally positive connection matrices. The other approach, which we call the "geometric approach," derives the points of the curves by successively cutting corners of a given polygonal arc. This approach owes its latest developments to the author [13, 14].

In Section 2 we consider the relatively simple case of piecewise cubic polynomials and this is extended to piecewise cubic rationals in Section 3. By concentrating on these cases it is hoped that the essential points of the general case can be illustrated without being obscured by the extra

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complexity of higher degree. Both approaches are discussed in more generality in Section 4. In Section 5 we consider the restriction required on the parameters in the geometric approach to ensure continuity of certain geometric properties of the curve, and discuss possible choices of parameters and their effect on the shape of the curve.

Although this paper is intended primarily as a survey and elucidation of results already known, all of Sections 2 to 5 contain material which has not appeared before and which it is hoped will prove useful both for understanding and applying the geometric approach. We finish by drawing some conclusions in Section 6.

2. Piecewise Cubics

In this section we consider the construction of a continuous piecewise cubic polynomial $S\colon \mathbb{R} \to \mathbb{R}^s$, $s \geqslant 2$. We are particularly interested in curves which look smooth. Following ideas first introduced in [16], we define the curve S to be *geometrically continuous of order* 2 (G^2) if it has continuous unit tangent and continuous curvature at all points where S' is non-zero. If S' is zero at a point, then the curve may have a cusp there, but we do not consider here conditions for ensuring S' is non-zero. We compare and contrast two approaches which we refer to as "analytic" and "geometric."

The Analytic Approach

By reparametrizing, if necessary, we can assume that the curve S has knots at the integer $\mathbb{Z}$, i.e., for each i in $\mathbb{Z}$, $S|[i, i+1]$ is a cubic polynomial. Then S is clearly G^2 for all $t \notin \mathbb{Z}$. For S to be G^2 at i in $\mathbb{Z}$ it can be shown to be necessary and sufficient that

$$(S'(i^+, S''(i^+))^{\mathrm{T}} = A_i(S'(i^-), S''(i^-))^{\mathrm{T}}, \tag{2.1}$$

where for some $\beta_i > 0$ and some γ_i,

$$A_i = \begin{bmatrix} \beta_i & 0 \\ \gamma_i & \beta_i^2 \end{bmatrix}. \tag{2.2}$$

The matrix A_i in (2.1) is called a connection matrix since it determines how the polynomial pieces of the curve connect together at $t = i$. If $\gamma_i \geqslant 0$ for all i in $\mathbb{Z}$, it has been shown that S can be written uniquely in the form

$$S(t) = \sum_{i=-\infty}^{\infty} P^i N_i(t), \tag{2.3}$$

where $P^i \in \mathbb{R}^s$ and $N_i\colon \mathbb{R} \to \mathbb{R}$ has support on $[i, i+4]$ with $\sum_{-\infty}^{\infty} N_i = 1$. Moreover it can be shown that for any strictly increasing sequence (t_i), the matrix $(N_j(t_i))_{i,j\in\mathbb{Z}}$ is totally positive, i.e., has all minors non-negative. This ensures that in some senses the shape of the curve S reflects the shape of the polygonal arc joining consecutive points P^i; see [12] for a discussion. For this reason the points P^i can be used to control the shape of the curve S and are therefore referred to as *control points*, and the polygonal arc joining them as the *control polygon*. It has been shown further that given any $j_1 < j_2 < \cdots < j_m$, any $m \geqslant 1$, then

$$\det(N_{j_k}(t_i))_{i,k=1}^{m} > 0 \Leftrightarrow j_i < t_i < j_i + 4, \qquad i = 1, ..., m. \tag{2.4}$$

In particular we see that

$$N_i(t) > 0 \Leftrightarrow i < t < i + 4. \tag{2.5}$$

Thus any minor of $(N_j(t_i))_{i,j\in\mathbb{Z}}$ is strictly positive if and only if its diagonal elements are non-zero and for arbitrary points $Q_1, ..., Q_m$ in $\mathbb{R}^s$, there is a unique curve $S = \sum_{j=1}^{m} P^j N_{i_j}$ satisfying the interpolation conditions

$$S(t_i) = Q_i, \qquad i = 1, ..., m,$$

if and only if

$$N_{j_i}(t_i) > 0, \qquad i = 1, ..., m.$$

In fact all of these above properties continue to hold if the connection matrix A_i has the form

$$A_i = \begin{bmatrix} \beta_i & 0 \\ \gamma_i & \delta_i \end{bmatrix}, \tag{2.6}$$

where $\beta_i > 0$, $\delta_i > 0$, $\gamma_i \geqslant 0$, though in this case, of course, S will not in general be G^2.

For a brief history, we remark that for (2.2) with $\gamma_i = 0$, the above theory reduces to the classical theory of B-splines, initiated by Schoenberg and developed by him, de Boor, and others, see [4]. In [2] Barsky considered the case (2.2) when β_i, γ_i are independent of i and the general case (2.6) was studied by the author in [11], where an explicit formula was given for N_i.

For the case of B-splines, the theory can be elegantly developed by using polar forms, see Ramshaw [17]. This approach has been extended to the general case by Seidel [20] and we describe this briefly here as it has some bearing on the geometric approach to be considered shortly. Take $u < v < w$, either in an interval $[i, i+1]$ or in an interval $[i-1, i+1]$ with $v = i$. We denote by $f(u, v, w)$ the intersection of the osculating planes to the curve S at the points $S(u)$, $S(v)$ and $S(w)$. To ensure this is well-defined we must assume that S is "non-singular" in some sense, but any "singular" curve can be gained as a projection of a non-singular one. Similarly $f(u, u, v)$ denotes the intersection of the tangent line at $S(u)$ with the osculating plane at $S(v)$, while $f(u, u, u) = S(u)$. Then it can be seen that the control point P^i equals $f(i+1, i+2, i+3)$. In general the function f, the *polar form* of S, is a piecewise rational function in each of its variables, but in the special case of B-splines, f is affine in each variable.

The Geometric Approach

In order to describe the geometric construction of a piecewise cubic polynomial curve, we must first describe the geometric construction of a single cubic polynomial segment in $\mathbb{R}^s$. By reparametrizing, if necessary, we may assume it is of the form

$$p(t) = \mathbf{b}_0(1-t)^3 + \mathbf{b}_1 3t(1-t)^2 + \mathbf{b}_2 3t^2(1-t) + \mathbf{b}_3 t^3, \qquad 0 \leqslant t \leqslant 1, \quad (2.7)$$

for points $\mathbf{b}_0, ..., \mathbf{b}_s$ in $\mathbb{R}^s$ called *Bézier points.* For each t, $0 \leqslant t \leqslant 1$, we can clearly construct $p(t)$ as illustrated in Fig. 1 where, as elsewhere, numbers against line segments denote ratios.

Thus to construct a piecewise cubic polynomial curve, we construct the Bézier points $\mathbf{b}_0^i, ..., \mathbf{b}_3^i$ for the ith segment p_i, each $i \in \mathbb{Z}$. We now give such a geometric construction which ensures that the curve is G^2. This is essentially due to Farin [6], though he described it somewhat differently. We start with arbitrary points $P^i \in \mathbb{R}^s$ and numbers $\beta_i, \lambda_i, \mu_i > 0$, $i \in \mathbb{Z}$, with

$$\lambda_i \mu_i = \beta_i^2. \quad (2.8)$$

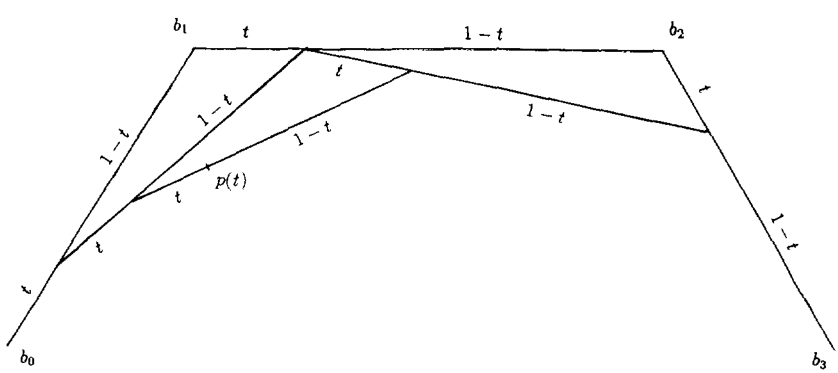

FIG. 1. Evaluating a cubic polynomial.

We then construct Bézier points $\mathbf{b}_0^i, \ldots, \mathbf{b}_3^i$, $i \in \mathbb{Z}$, as illustrated in Fig. 2. The whole curve $S\colon \mathbb{R} \to \mathbb{R}^s$ can then be given by

$$S(t) = p_i(t-1), \qquad i \leqslant t \leqslant i+1, \quad i \in \mathbb{Z}. \tag{2.9}$$

Since $S(t)$, $i \leqslant t \leqslant i+1$, is a convex combination of $P^{i-3}, \ldots, P^i$, we have

$$S(t) = \sum_{i=-\infty}^{\infty} P^i N_i(t), \tag{2.3}$$

where $N_i\colon \mathbb{R} \to \mathbb{R}$ is positive with the support on $[i, i+4]$ and $\sum_{-\infty}^{\infty} N_i = 1$. Moreover we shall see more generally in Section 4 that all the total positivity properties of the analytic approach hold also in this case. Indeed they hold for any $\beta_i, \lambda_i, \mu_i > 0$, regardless of (2.8), though without (2.8) the curve will not in general be G^2.

What, then, is the precise relationship between the two approaches? We shall see this by calculating the connection matrix A_i for (2.3) in terms of β_i, λ_i, and μ_i. Recalling (2.7), (2.9) we have for $j = 0, 1, 2$,

$$\begin{aligned} S^{(j)}(i^-) &= p_{i-1}^{(j)}(1^-) = j!\, \Delta^j \mathrm{b}_3^{i-1}, \\ S^{(j)}(i^+) &= p_i^{(j)}(0^+) = j!\, \Delta^j b_j^i, \end{aligned} \tag{2.10}$$

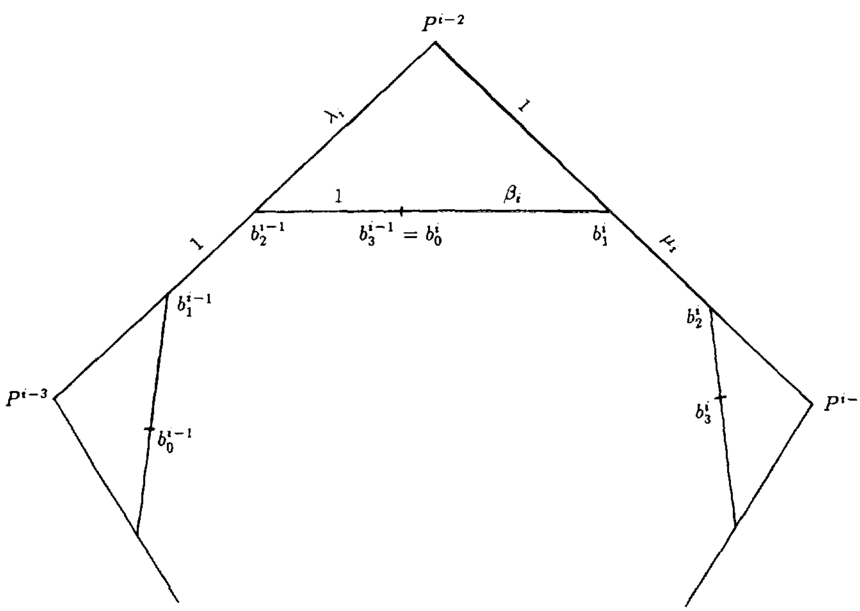

FIG. 2. Constructing a piecewise cubic polynomial.

where Δ is the backward difference

$$\Delta a_i = a_i - a_{i-1}.$$

From the geometric construction a straightforward calculation gives

$$\begin{aligned} \Delta \mathbf{b}_1^i &= \beta_i \Delta \mathbf{b}_3^{i-1}, \\ \Delta^2 \mathbf{b}_2^i &= \lambda_i \mu_i \Delta^2 \mathbf{b}_3^{i-1} + (\mu_i + \beta_i \mu_i - \lambda_i \mu_i - \beta_i) \Delta \mathbf{b}_3^{i-1}. \end{aligned} \tag{2.11}$$

Combining (2.10) and (2.11) gives

$$(S'(i^+), S''(i^+))^{\mathrm{T}} = A_{i+2}(S'(i^-), S''(i^-))^{\mathrm{T}},$$

where

$$A_i = \begin{bmatrix} \beta_i & 0 \\ 2(\mu_i + \beta_i \mu_i - \lambda_i \mu_i - \beta_i) & \lambda_i \mu_i \end{bmatrix}.$$

Thus A_i can be any matrix of the form

$$A_i = \begin{bmatrix} \beta_i & 0 \\ \gamma_i & \delta_i \end{bmatrix}$$

for $\beta_i > 0$, $\delta_i > 0$, and $\gamma_i > -2(\beta_i + \delta_i)$.

To ensure a G^2 curve we take $\delta_i = \beta_i^2$ and $\gamma_i > -2\beta_i(1+\beta_i)$. Thus the geometric construction is considerably more general than the analytic approach, which requires $\gamma_i \geqslant 0$.

We shall say that the control vertices P^i, $i \in \mathbb{Z}$, lie in *general position* if for any i, $P^i, \ldots, P^{i+3}$ do not lie in a common plane. In this case we adopt the following labelling scheme which is crucial for generalization to higher degree. We denote by i^3 the point $S(i) = \mathbf{b}_3^{i-1} = \mathbf{b}_0^i$, and by i^2 and i respectively the tangent line and osculating plane to S at this point. Other labels of consecutive integers denote the intersections of their components; e.g., $1^2 2$ denotes the point of intersection of the line 1^2 and the plane 2. In Fig. 3 we use this scheme to label the points and lines of Fig. 2, with $i = 2$.

This labelling scheme is closely related to the polar form for the analytic approach. Indeed we describe briefly how the polar form can be defined for the geometric approach. Take t, $i < t < i+1$, and denote by t^3, t^2, t respectively the point $S(t)$ and the tangent line and osculating plane at this point. As before we define labels $(i-1)\,it$, $it(i+1)$, and $t(i+1)(i+2)$ as intersections. Then it is easily seen that they are points and that the vertices $\ldots P^{i-3}, (i-1)\,it, it(i+1), t(i+1)(i+2), P^i$, are in general position. From these new vertices we can construct the curve S as before, but now the polynomial segment p_i is split into two segments, one on $[i, t]$ and one on

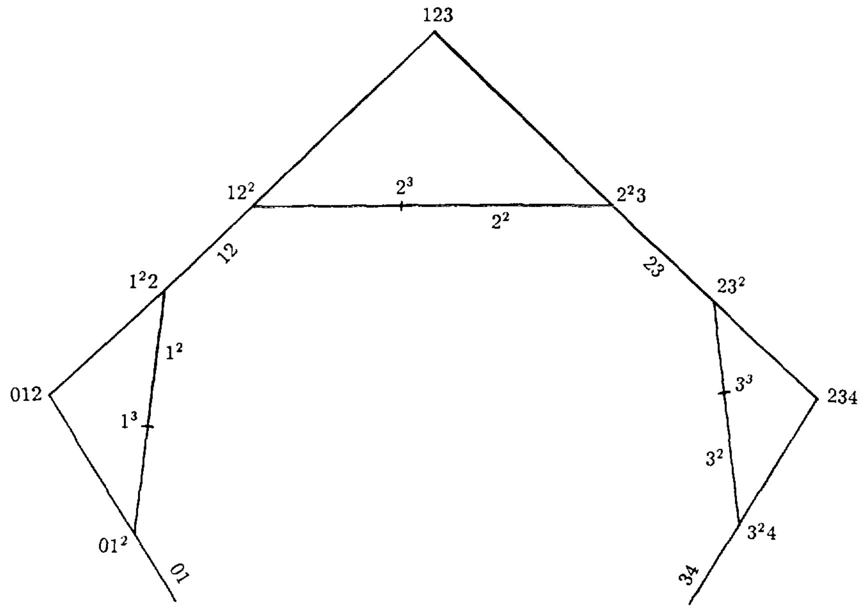

FIG. 3. Labelling for a piecewise cubic polynomial.

$[t, i+1]$. This allows us to define points with labels i^2t, $t(i+1)^2$, it^2 and $t^2(i+1)$. By repeating this procedure we can successively introduce any new values of the parameter. So we can define $f(u, v, w) = uvw$, $f(u, u, v) = u^2v$, etc., to give a polar form as in the analytic approach.

3. Piecewise Cubic Rationals

The use of piecewise rational curves offers the significant advantages over piecewise polynomials of extra flexibility and invariance under projection (as we describe below). Moreover, we shall see that the theory for rationals is almost as simple as that for polynomials. Indeed, the construction of piecewise rationals in projective space is in a sense the natural setting for the geometric approach, though we only mention this briefly. In considering piecewise rationals, the analytic and geometric approaches diverge further.

The Analytic Approach

Consider a continuous piecewise cubic polynomial curve $S: \mathbb{R} \to \mathbb{R}^{s+1}$ $s \geqslant 2$, given by

$$S(t) = \sum_{i=-\infty}^{\infty} (P^i w_i, w_i)\, N_i(t), \tag{3.1}$$

for $P^i \in \mathbb{R}^s$, $w_i > 0$, $i \in \mathbb{Z}$, and with connection matrices given by (2.6). If $P: \mathbb{R}^{s+1} \to \mathbb{R}^s$ denotes the projection

$$P(x_1, ..., x_{s+1}) = \left(\frac{x_1}{x_{s+1}}, ..., \frac{x_s}{x_{s+1}}\right), \tag{3.2}$$

we can define a piecewise cubic rational curve $PS: \mathbb{R} \to \mathbb{R}^s$ by

$$PS(t) = \frac{\sum_{i=-\infty}^{\infty} P^i w_i N_i(t)}{\sum_{i=-\infty}^{\infty} w_i N_i(t)}. \tag{3.3}$$

A straightforward calculation shows that at i, PS has connection matrix

$$B_i = \begin{bmatrix} \beta_i & 0 \\ v_i & \delta_i \end{bmatrix}, \tag{3.4}$$

where

$$v_i = \gamma_i + \frac{2w'(i\ )}{w(i)}(\delta_i - \beta_i^2), \qquad w = \sum_{i=-\infty}^{\infty} w_i N_i. \tag{3.5}$$

Thus B_i is independant of the weights w_i if and only if $\delta_i = \beta_i^2$, which we recall from Section 2 ensures that S is G^2. This is also the condition for S and PS to have the same connection matrix. These properties are considered more generally in [9, 10].

We may rewrite (3.3) as

$$PS(t) = \sum_{i=-\infty}^{\infty} P^i \hat{N}_i(t), \qquad \hat{N}_i = \frac{w_i N_i}{w}. \tag{3.6}$$

Then $\sum_{i=-\infty}^{\infty} \hat{N}_i = 1$ and

$$\det(\hat{N}_{j_k}(t_i))_{i,k=1}^m = \det(N_{j_k}(t_i))_{i,k=1}^m \frac{w_{j_1} ... w_{j_m}}{w(t_1) ... w(t_m)}.$$

So the total positivity properties of the basis functions N_i immediately imply corresponding properties for the functions $\hat{N}_i$. We remark that the connection condition (2.1) for S can be relaxed to allow dependence also on $S(i)$, while still retaining all the above properties for PS.

The Geometric Approach

We first describe the geometric construction of a single cubic rational segment in $\mathbb{R}^s$. The segment is determined by Bézier points $\mathbf{b}_0, ..., \mathbf{b}_3$ in $\mathbb{R}^s$ and for $i = 0, 1, 2$, by an *auxiliary point* Q_i lying strictly within the line segment joining $\mathbf{b}_i$ to $\mathbf{b}_{i+1}$. For collinear points A, B, C, D we use the notation $[A, B, C]$ for the ratio in which B divides the segment AC and use the notation (A, B, C, D) for the cross-ratio $[A, C, B][B, D, A]$.

Take $0 < t < 1$. For $i = 0, 1, 2$, we define $\mathbf{b}_i^{(1)}$ on the line segment $\mathbf{b}_i\mathbf{b}_{i+1}$ so that

$$(\mathbf{b}_i, \mathbf{b}_{i+1}, \mathbf{b}_i^{(1)}, Q_i) = \frac{t}{1-t}. \tag{3.7}$$

For $i = 0, 1$, we define $\mathbf{b}_i^{(2)}$ to be the intersection of the line $\mathbf{b}_i^{(1)}\mathbf{b}_{i+1}^{(1)}$ with the line joining $\mathbf{b}_{i+1}$ to the point of intersection of the lines $\mathbf{b}_i\mathbf{b}_{i+1}^{(1)}$ and $\mathbf{b}_i^{(1)}\mathbf{b}_{i+2}$, see Fig. 4. Finally we define $r(t)$ as the intersection of $\mathbf{b}_0^{(2)}\mathbf{b}_1^{(2)}$ with the line joining $\mathbf{b}_1^{(1)}$ to the point of intersection of $\mathbf{b}_0^{(1)}\mathbf{b}_1^{(2)}$ and $\mathbf{b}_0^{(2)}\mathbf{b}_2^{(1)}$.

If we write

$$[\mathbf{b}_1, Q_i, \mathbf{b}_{i+1}] = \frac{w_i + 1}{w_i}, \qquad i = 0, 1, 2, \tag{3.8}$$

for positive *weights* $w_0, ..., w_3$, then it can be shown that

$$r(t) = \frac{\mathbf{b}_0 w_0(1-t)^3 + \mathbf{b}_1 w_1 3t(1-t)^2 + \mathbf{b}_2 w_2 3t^2(1-t) + \mathbf{b}_3 w_3 t^3}{w_0(1-t)^3 + w_1 3t(1-t)^2 + w_2 3t^2(1-t) + w_3 t^3}, \qquad 0 < t < 1. \tag{3.9}$$

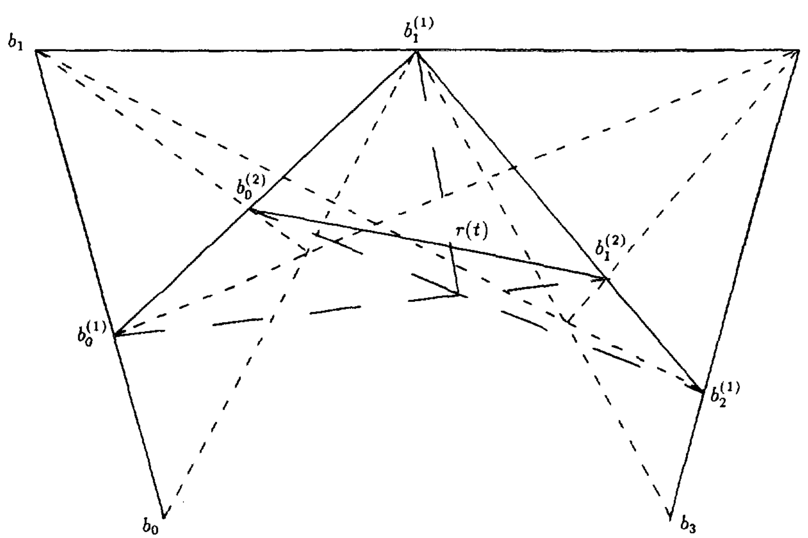

FIG. 4. Recursive evaluation for a rational cubic.

The above construction appears in [14] and is a modification of a construction due to Farin [7]. It is clear that the construction is projective invariant; i.e., if P is a projection from $\mathbb{R}^s$ to $\mathbb{R}^t$, $t \leqslant s$, then the curve Pr has Bézier points $P\mathbf{b}_0, ..., P\mathbf{b}_3$ and auxiliary points $PQ_0, ..., PQ_2$. The construction thus makes sense in projective space and the theory in [14] is largely developed in this setting. However, in this paper we are interested in affine representations of the form (2.3), the total positivity properties of the basis functions N_i and the connection with the analytic approach, and so we consider only curves in $\mathbb{R}^s$.

We note that by scaling top and bottom of (3.9) and making a linear/linear change of parameter, the curve r depends, for given $\mathbf{b}_0, ..., \mathbf{b}_3$, only on $w_0 w_2/w_1^2$ and $w_1 w_3/w_2^2$. So we could, for example, assume $w_0 = w_3 = 1$.

A piecewise rational curve can now be formed from a sequence of control points P^i by constructing Bézier points of successive segments in precisely the manner of the geometric approach of Section 2. The only difference is that to obtain G^2 continuity the equation (2.8) must be replaced by

$$\frac{w_0^{i+1} w_2^{i+1}}{(w_1^{i+1})^2} \frac{(w_2^i)^2}{w_1^i w_3^i} \lambda_i \mu_i = \beta_i^2, \tag{3.10}$$

where $w_0^i, ..., w_3^i$ denote the weights for the segment with Bézier point $\mathbf{b}_0^i, ..., \mathbf{b}_3^i$. Again the resulting curve has the form (2.3), where $N_i : \mathbb{R} \to \mathbb{R}$ has support on $[i, i+4]$ with $\sum_{-\infty}^{\infty} N_i = 1$ and all the total positiveness properties are preserved.

To conclude this section we make some comparisons between the two approaches for constructing piecewise rational curves. The analytic approach uses basis functions N_i which are piecewise polynomials and constructs curves of form (3.3). The geometric approach uses basis functions N_i which are piecewise rationals and constructs curves of form (2.3). Both construct an affine theory by fixing the denominator independent of the control points P^i. However in the analytic approach, the denominator depends on one weight w_i per segment, while in the geometric approach it depends on two weights w_1^i, w_2^i per segment (putting $w_0^i = w_3^i = 1$), and hence the geometric approach provides more flexibility.

4. The General Case

Having dealt in some detail with the construction of continuous piecewise cubic rational curves we briefly describe how this extends to arbitrary degree n, $n \geqslant 1$. As before we may assume knots at $\mathbb{Z}$ and so the curve is of form $S: \mathbb{R} \to \mathbb{R}^s$, $s \geqslant 2$, where $S|[i, i+1]$ is a rational function with top and bottom of degree n. We assume that for each $i \in \mathbb{Z}$,

$$(S'(i^+), ..., S^{(n-1)}(i^+))^{\mathrm{T}} = A_i(S'(i^-), ..., S^{(n-1)}(i^-))^{\mathrm{T}}$$

for some connection matrix A_i which is lower triangular and non-singular. We postpone until later a discussion of the relationship between A_i and continuity of geometric properties of the curve S at i.

The Analytic Approach

First consider the case where $S|[i, i+1]$ is a polynomial of degree n for each $i \in \mathbb{Z}$. If each A_i is diagonal with entries $(A_i)_{jj} = \beta_i^j$, $j = 1, ..., n-1$, some $\beta_i > 0$, the construction of S reduces to the classical theory of constructing spline functions in terms of B-splines, see [4]. For A_i positive and one-banded, i.e., $(A_i)_{jk} = 0$ unless $j = k$ or $k+1$, the theory was given in [11]. This was extended to general totally positive matrices (lower triangular and non-singular) in [5]. For this case we can express S in the form

$$S(t) = \sum_{i=-\infty}^{\infty} P^i N_i(t), \tag{2.3}$$

where $P^i \in \mathbb{R}^s$ and $N_i: \mathbb{R} \to \mathbb{R}$ is positive with $N_i(t) > 0$ if and only if $i < t < i+n+1$, and $\sum_{-\infty}^{\infty} N_i = 1$. As for the case $n = 3$, for any strictly increasing sequence (t_i) the matrix $(N_j(t_i))_{i,j \in \mathbb{Z}}$ is totally positive and any minor is strictly positive if and only if its diagonal elements are non-zero.

The rational case is dealt with as for $n = 3$ by considering $S: \mathbb{R} \to \mathbb{R}^{s+1}$ of form (3.1) for connection matrices A_i and then defining a piecewise rational curve $PS: \mathbb{R} \to \mathbb{R}^s$ by (3.3). This can then be written in the form (3.6) and all the total positivity properties go over to the basis functions $\hat{N}_i$. The relationship between the connection matrices A_i for S and the connection matrices for PS is discussed in [9, 10].

The Geometric Approach

For a single rational segment we can extend (3.9) in an obvious way as

$$r(t) = \frac{\sum_{i=0}^{n} \mathbf{b}_i w_i \binom{n}{i} t^i(1-t)^{n-i}}{\sum_{i=0}^{n} w_i \binom{n}{i} t^i(1-t)^{n-i}}, \qquad 0 \leqslant t \leqslant 1,$$

for Bézier points $\mathbf{b}_i \in \mathbb{R}^s$ and weights $w_i > 0$. As for the cubic case there is no loss of generality in assuming $w_0 = w_n = 1$. Defining auxiliary points $Q_0, ..., Q_{n-1}$ by

$$[\mathbf{b}_i, Q_i, \mathbf{b}_{i+1}] = \frac{w_{i+1}}{w_i},$$

the segment is represented in a projective invariant manner by the Bézier points and auxiliary points. The geometric construction of $r(t)$, $0 < t < 1$, from these points extends that of the cubic case in an obvious way, see [14], and we do not describe it again.

The geometric construction of piecewise rational cubics was extended to the quartic case by Boehm [3]. This was simplified and extended to general degree by the author [13]. As in the cubic case, we start with a sequence of control points P^i, $i \in \mathbb{Z}$, and recursively construct new points until we reach the Bézier points $\mathbf{b}_0^i, ..., \mathbf{b}_n^i$ for the ith segment, $i \in \mathbb{Z}$. The form of this construction is the same for piecewise rationals as for piecewise polynomials.

A precise description of the construction in the general case requires a generalization of the labelling scheme for the case $n = 3$ described in Section 2. However this description is somewhat involved and we shall not give it here, it being described in [13, 14]. Here we remark that each step in the construction is of one of the following three types.

Type 1. Given two points A and B, we insert a new point P on the line segment AB so that $[A, P, B] = \lambda^{-1}$ for some parameter $\lambda > 0$.

Type 2. Given two points A and B, we insert points P and Q on the line segment AB so that $[A, P, Q] = \lambda^{-1}$, $[P, Q, B] = \mu^{-1}$, for parameters $\lambda > 0$, $\mu > 0$.

Type 3. Given points A, B, C, we insert points P and Q on the segments AB and BC respectively, so that $[A, P, B] = \lambda^{-1}$, $[B, Q, C] = \mu^{-1}$ for some parameters $\lambda > 0$, $\mu > 0$. We then define R to be in the intersection of the line segments AQ and PC.

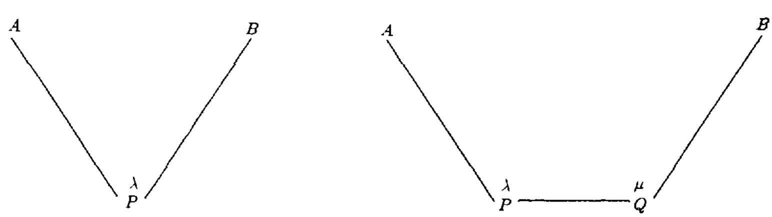

FIG. 5. Schematic representation of corner cutting.

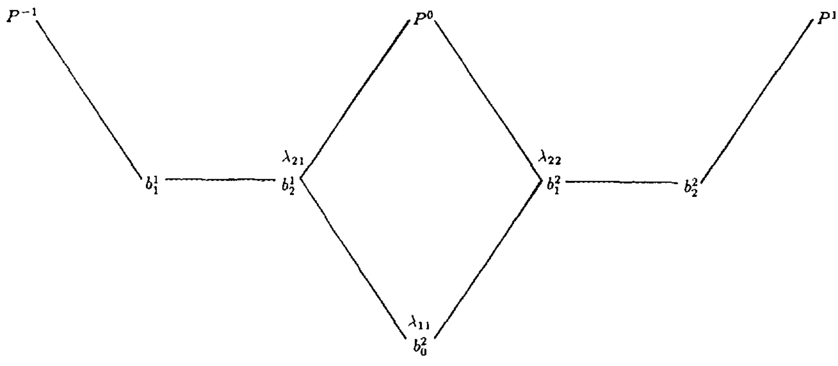

FIG. 6. Geometric construction for $n = 3$.

Starting with the control points $P^i = P^i_{1,0}$, we construct points $P^i_{k,j}$, $k = 2, \ldots, n$, $j = 0, \ldots, k-1$, recursively on k to finally gain the Bézier points

$$\mathbf{b}^i_j = P^i_{n,j}, j = 0, \ldots, n-1, \mathbf{b}^i_n = \mathbf{b}^{i+1}_0.$$

For each k, the points $P^i_{k,j}$, $i \in \mathbb{Z}, j = 0, \ldots, k-1$, can be thought of as control points for a curve with knots of multiplicity k; i.e., the rational

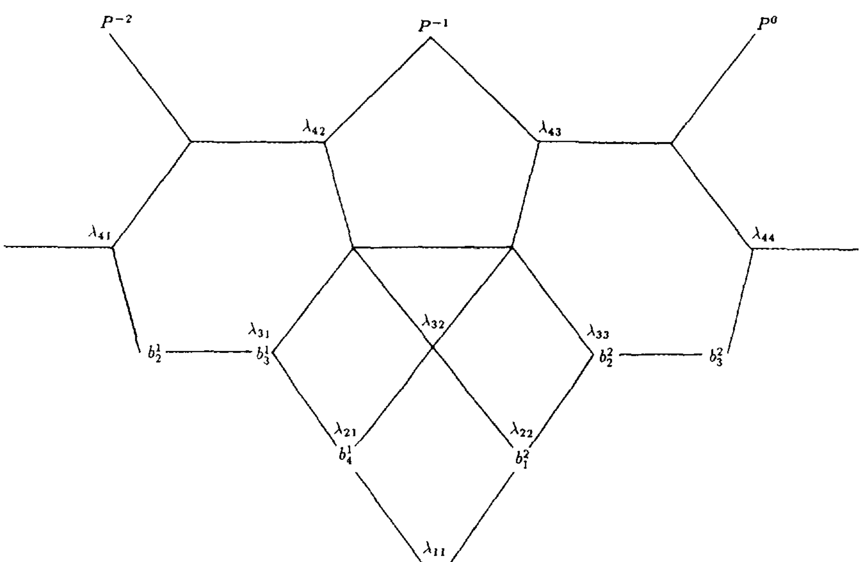

FIG. 7. Geometric construction for $n = 5$.

segments join with a linear relationship between their derivatives up to order $n-k$. To get from the points $P^i_{i,j}$, $i \in \mathbb{Z}$, $j=0, ..., k-1$, to the points $P^i_{k+1,j}$, $i \in \mathbb{Z}$, $j=0, ..., k$, requires $n-k$ parameters per segment: $\lambda^i_{n-k,j}$, $j=1, ..., n-k$.

We illustrate schematically the constructions for $n=3, 5$, and 7 in Figs. 6, 7, and 8 respectively, where we represent constructions of types 1 and 2 as in Fig. 5. To avoid congestion we label only the parameters $\lambda^2_{n-k,j}$ and omit the superscript 2. Of course the case $n=3$ was described geometrically in Section 2. We ignore the cases $n=4$ and 6 partly because

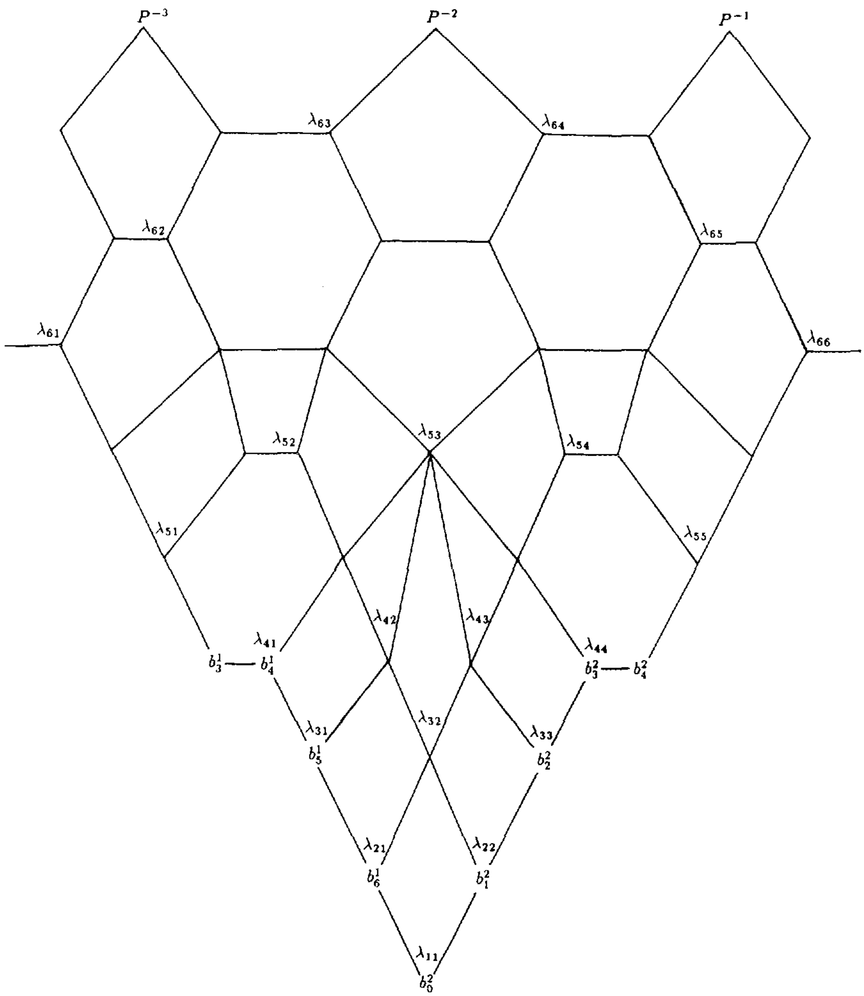

FIG. 8. Geometric construction for $n=7$.

this avoids using constructions of type 3 and partly because curves of odd degree allow more helpful manipulation of shape by the parameters. The schematic illustrations are sufficient to allow practical implementation of the constructions.

As for the cubic case we can give the resulting curve a global parametrization $S: \mathbb{R} \to \mathbb{R}^s$ so that $S(t)$, $i \leqslant t \leqslant i+1$, is a convex combination of $P^{i-n}, ..., P^i$ and so can be expressed in the form (2.3), where $N_i: \mathbb{R} \to \mathbb{R}$ is positive with support on $[i, i+n+1]$ and $\sum_{-\infty}^{\infty} N_i = 1$. We shall now see that the same total positivity properties hold as for the analytic approach.

Let $\mathbf{b}$ denote the vector $(\mathbf{b}_0^i, ..., \mathbf{b}_n^i, \mathbf{b}_1^{i+1}, ..., \mathbf{b}_n^{i+1}, ..., \mathbf{b}_1^j, ..., \mathbf{b}_n^j)^{\mathrm{T}}$ for some $i \leqslant j$. In the geometric construction these Bézier points are derived from a vector of control points $P = (P^k, P^{k+1}, ..., P^l)^{\mathrm{T}}$ for some $k < l$. Each step in the geometric construction is equivalent to constructing a new vector of points by applying a positive, one-banded matrix to the previously constructed vector of points; see [12]. Since a product of one-banded, positive matrices is totally positive, we see that $\mathbf{b} = AP$, where A is totally positive.

Given any sequence of parameter values $t_1 < \cdots < t_m$, the vector $T = (S(t_1), ..., S(t_m))^{\mathrm{T}}$ is gained from a sequence $\mathbf{b}$ of Bézier points in the same manner as above and thus $T = B\mathbf{b}$ for a totally positive matrix B. So we have $T = CP$, where $C = BA$ is totally positive. But from (2.3) we know that $C = (N_j(t_i))_{i=1}^{m}{}_{j=k}^{l}$. Since the choice of parameter values t_i was arbitrary we know that for any strictly increasing sequence (t_i) the matrix $M = (N_j(t_i))_{i,j \in \mathbb{Z}}$ is totally positive.

Moreover, which minors of a one-banded positive matrix are strictly positive depends only on which entries are strictly positive. It thus follows that which minors of M are strictly positive is independent of the values of the parameters $\lambda_{j,k}^i$ (assuming of course that they are strictly positive). But appropriate choice of parameters gives us the B-spline case for which we know that

$$\det(M_{i,j_k})_{i,k=1}^m = \det(N_{j_k}(t_i))_{i,k=1}^m > 0 \Leftrightarrow i_j < t_j < i_j + n + 1$$

and so this result holds in the general case also.

5. Smoothness

In Section 4 we did not consider the smoothness of the constructed curve S. For $n = 3$ we considered a choice of matrices A_i which ensured that S was G^2. The concept of G^2 continuity has been extended in two different ways. We say S is geometrically continuous of order m (G^m) if whenever S'

is non-zero, S is C^m with respect to arc length. The conditions on A_i to ensure G^m continuity are given, for example, in [8, 11], but they are somewhat involved and so to derive a simpler, more general theory we consider the weaker condition that A_i has diagonal elements $(A_i)_{jj} = \beta_i^j$, $j = 1, ..., n-1$, for some $\beta_i > 0$. There is no restriction on the entries $(A_i)_{jk} = 0$ for $j > k$. (Recall that $(A_i)_{jk} = 0$ for $j < k$.) In this case we say S has Frenet frame continuity of order $n-1$, or is F^{n-1}. The reason for this terminology is that it was shown by Dyn and Micchelli [5] that this condition is equivalent to the continuity of the Frenet frame and associated $n-2$ curvatures (see also [15]), provided of course that the Frenet frame is well-defined, i.e., $S', ..., S^{(n-1)}$ are linearly independent at the point. For example, F^3 continuity is equivalent to G^2 continuity plus continuity of the torsion.

In [13] are given necessary and sufficient conditions on the parameters $\lambda_{k,j}^i$, $k = 1, ..., n-1, j = 1, ..., k$ for the curve S to be F^{n-1} continuous. They are of the form

$$w_k^{i+1} \lambda_{k,1}^i \cdots \lambda_{k,k}^i R_k^i = \beta_i^k w_{n-k}^i, \qquad k = 1, ..., n-1 \tag{5.1}$$

where R_k^i is a simple rational function of certain parameters $\lambda_{l,j}^i$ for $l < k$, which equals 1 when $k \leqslant \frac{1}{2}(n+1)$. (Here we have assumed $w_0^{i+1} = w_n^i = 1$.) For example, the conditions for $n = 5$ have $R_1^i = R_2^i = R_3^i = 1$,

$$R_4^i = \frac{(1 + (\lambda_{22}^{i-1})^{-1})}{(1 + \lambda_{21}^{i+1})} \frac{(1 + (\lambda_{32}^{i-1})^{-1})}{(1 + \lambda_{32}^{i+1})}. \tag{5.2}$$

For the case of piecewise polynomials of arbitrary degree n, when all the weights equal 1, the condition (5.1) can be satisfied by taking the following choices. We take $\beta_i = 1$ and for $k \leqslant n-2$ we assume $\lambda_{k,j}^i = \lambda_{k,j}$, independent of i, where

$$\lambda_{k,j} \lambda_{k,k-j+1} = 1, \qquad j = 1, ..., k.$$

Under these assumptions, all the terms R_k^i equal 1 and (5.1) is automatically satisfied for $k = 1, ..., n-2$. The only remaining condition is

$$\lambda_{n-1,1}^i \cdots \lambda_{n-1,n-1}^i = 1. \tag{5.3}$$

Thus we have arbitrary global parameters $\lambda_{k,j}$, $k = 1, ..., n-2$, $1 \leqslant j \leqslant \frac{1}{2}k$, and local parameters $\lambda_{n-1,1}^i, ..., \lambda_{n-1,n-1}^i$, $i \in \mathbb{Z}$, subject only to the restriction (5.3).

The parameters $\lambda_{k,j}^i$ can be used to manipulate the shape of the curve S without altering the control points P^i and the effect of altering the parameters can sometimes be deduced from the form of the geometric construction. We illustrate this with the case of a piecewise quintic polynomial curve. We assume $\beta_i = \lambda^i_{11} = \lambda^i_{32} = \lambda^i_{42} = \lambda^i_{43} = 1$, $\lambda^i_{21} = (\lambda^i_{22})^{-1} = \alpha$, $\lambda^i_{31} = (\lambda^i_{33})^{-1} = \gamma_i$ and $\lambda^i_{41} = (\lambda^i_{44})^{-1} = \delta_i$, for the parameters $\alpha, \gamma_i, \delta_i, i \in \mathbb{Z}$. We see from (5.2) that $R^i_4 = 1$ and hence that the conditions (5.1) for F^4 continuity are all satisfied.

Now letting the global parameters α diverge to ∞ makes b^{i-1}_4 approach b^{i-1}_3 and b^i_1 approach b^i_2 and thus the curvature at b^i_0 approaches zero, $i \in \mathbb{Z}$. In contrast, letting $\alpha \to 0$ causes b^{i-1}_4 and b^i_1 to approach b^i_0 and so the curve will approach a sharp corner at b^i_0, $i \in \mathbb{Z}$. Letting $\gamma_i \to 0$ causes the points $b^{i-1}_3, b^{i-1}_4, b^i_1, b^i_2$ to approach collinearity and hence the curvature at b^i_0 to approach zero. If both $\gamma_i \to \infty$ and $\alpha \to \infty$, then both the curvature and torsion at b^i_0 will approach zero. Decreasing δ_i will have the effect of pulling the curve near b^i_0 towards the control vertex P^{i-3} while if $\delta_i \to 0$ and $\gamma_i \to 0$, then b^i_0 will converge to P^{i-3} and hence the curve will be pulled into the corner of the control polygon at P^{i-3}. If $\delta_i \to 0$ and $\gamma_i \to 0$ for all i, then the curve will approach the control polygon.

6. Conclusions

We have discussed a method of constructing piecewise rational curves by successively cutting corners of a given control polygon until we reach the Bézier points for the rational segments. This "geometric approach" has been compared with the "analytic approach" of projecting a piecewise polynomial curve whose pieces are joined by totally positive connection matrices. The geometric approach has the advantage of being in some senses more general and of giving very simple algorithms for evaluation with parameters which may have clear geometric significance. However, the analytic approach has an elegant theory which allows knots of varying multiplicity and hence general knot insertion, as well as formulas for representing polynomials [1]. It would be interesting to elucidate further the connection between these two approaches; for example, to determine an easily definable class of connection matrices allowed by the geometric approach which is more general than that of totally positive matrices.

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